Math 10A
Worksheet, Discussion #28; Wednesday, 7/25/2018
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1 First-Order ODEs

1.1 Concepts

1. Integrating factors is for **linear, first-order** differential equations. It is of the form

$$\boxed{y' + P(t)y = Q(t).}$$

To solve this, we multiply the equation by the **integrating factor** $I(t) = e^{\int P(t)dt}$. Then we can calculate that $I'(t) = I(t)P(t)$ and

$$(I(t)y)' = I(t)y' + I'(t)y = I(t)y' + I(t)P(t)y = I(t)(y' + P(t)y) = I(t)Q(t).$$

Taking the integral gives us

$$I(t)y = \int I(t)Q(t)dt \implies y = \frac{1}{I(t)}\int I(t)Q(t)dt.$$

2. Separable equations is for **first-order** differential equations which can be separated. It is **separable** if we can write it of the form

$$\boxed{y' = f(y)g(t).}$$

We write $y' = \frac{dy}{dt}$ and split it to get

$$\int \frac{dy}{f(y)} = \int g(t)dt.$$

If $f(y)$ is a polynomial, we use **partial fractions** to integrate the left side. There are also solutions gotten by solving $f(y) = 0$.

1.2 Problems

3. True **FALSE** We cannot use the method of separable equations on $y' = e^{y+t}$ because it involves a sum of y and t.

> **Solution:** We actually have $e^{y+t} = e^y e^t$ and hence it is the product of functions involving t and y and hence we can use separable equations.

4. True **FALSE** If we can use the method of separable equations, we must be able to write $y' = (ay + b)f(t)$ for a linear polynomial in terms of y.

Solution: We do not need the function in y to be linear.

5. True **FALSE** The equation $y' = y + t$ is not separable and so we do not know how to solve it.

Solution: It is indeed not separable but we can write it as $y' - y = t$ which is a form that we know how to solve with integrating factors.

6. Find the solution of $y' + 2xy = 2x$ with $y(0) = 0$.

Solution: The integrating factor is $e^{\int 2x dx} = e^{x^2}$. Multiplying by this gives

$$e^{x^2}y' + 2xe^{x^2}y = (e^{x^2}y)' = 2xe^{x^2}.$$

Now integrating gives $e^{x^2}y = e^{x^2} + C$ and hence $y = 1 + Ce^{-x^2}$. Plugging in the initial condition gives us that $0 = 1 + Ce^{-0} = 1 + C$ and hence $C = -1$ so the solution is $y = 1 - e^{-x^2}$.

7. Find the general solution to $y' - \frac{y}{x+1} = (x + 1)^2$.

Solution: The integrating factor is $I(x) = e^{\int -\frac{1}{x+1} dx} = e^{-\ln(x+1)} = \frac{1}{x+1}$. We multiply through by this to get

$$\frac{y'}{x+1} - \frac{y}{(x+1)^2} = \left(\frac{y}{x+1}\right)' = x + 1.$$

Now, we integrate to get

$$\frac{y}{x+1} = \frac{(x+1)^2}{2} + C$$

and hence $y = \frac{(x+1)^3}{2} + C(x + 1)$.

8. Find the solution to $\frac{dy}{dt} = 2y + 3$ with $y(0) = 0$.

Solution: Separate to get

$$\frac{dy}{2y+3} = dt \implies \frac{\ln(2y+3)}{2} = t + C,$$

so $2y+3 = e^{2t+C}$. We can bring the constant down as multiplying by a constant so $e^{2t+C} = e^{2t} \cdot e^C = Ce^{2t}$ and so

$$y = \frac{Ce^{2t} - 3}{2}.$$

Taking $y(0) = 0$ gives $0 = (C-3)/2$ so $C = 3$ and we get

$$y = \frac{3e^{2t} - 3}{2}.$$

9. Find the solution to $y'e^y = 2t + 1$ with $y(1) = 0$.

Solution: We can rewrite this as $y' = (2t+1)e^{-y}$ and see this that is a separable equation because it is something involving t multiplied by something involving y. Now we can write this is $\frac{dy}{dt}e^y = 2t + 1$ and multiplying by dt gives

$$e^y dy = (2t+1)dt$$

and integrating gives $e^y = t^2 + t + C$ and this is the implicit solution. Now to find C, we plug in the initial condition to get $1 = e^0 = 1^2 + 1 + C = 2 + C$. Hence $C = -1$ and the solution is $e^y = t^2 + t - 1$.

10. Solve the IVP $y' = te^t$ with $y(0) = 0$.

Solution: We solve by integrating te^t. We do this using integration by parts. Let $u = t$ and $dv = e^t dt$ so $du = dt$ and $v = e^t$. Integrating gives $\int te^t dt = te^t - \int e^t dt = te^t - e^t + C$. Now we solve for our initial condition that $y(0) = 0e^0 - e^0 + C = C - 1 = 0$ so $C = 1$. Thus the solution is $te^t - e^t + 1$.

11. Solve the IVP $y' = \frac{1}{t \ln t}$ with $y(e) = 0$.

Solution: We find the integral using u substitution. Let $u = \ln t$ and then $y = \int \frac{1}{t \ln t} dt = \frac{du}{u} = \ln|u| + C + \ln(\ln t) + C$. Plug in our initial condition that $y(e) = 0$ to get $\ln(\ln e) + C = \ln 1 + C = C = 0$ so $y = \ln(\ln t)$ is our solution.